An Introduction to Determining Controllability using Forms

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May 14, 2007

Abstract:
This paper serves as a brief overview into determining controllability using forms, using the concept of the derived flag. It is not original work done by me, but it is an attempt to present the material from (cite source) in terms that someone not intimately familiar with differential geometry or exterior differential systems can understand clearly. The example of a penny rolling on a plane will be used to demonstrate the method of determining controllability using forms, as well as to show the similarities between the forms method and the more common method of using vector fields (the Lie Bracket method). There will also be a brief review of controllability using Lie Brackets, to make sure that everyone is up to speed with both methods.

Review of Controllability using Vector Fields:
This portion of the paper serves as a brief review of controllability using vector fields, as explained in lecture for ME598 – Geometric Mechanics. In order to determine whether a system is controllable using vector fields, one must be familiar with determining Lie Brackets. The Lie Bracket between two functions f(q) and g(q) is given by:

$$[f,g]=\frac{\partial g}{\partial q^i}f-\frac{\partial f}{\partial q^i}g \quad \textbf{(1)}$$

Consider a drift-less control affine system (the system is drift-less because if no control input is given, the system will remain at rest) of the form:

$$\dot{q}=g_1(q)u^1+\ldots+g_n(q)u^n \quad \textbf{(2)}$$

This system is controllable if:

$$span\{g_i,[g_i,g_j],[[g_i,g_j],g_k],\ldots\}=\mathbb{R}^n \quad \textbf{(3)}$$

Basically, the system is controllable if the span of the input vector fields g, along with the specified combination of Lie Bracket vector fields, span R^n. A more visual description of this is shown in Figure 1.

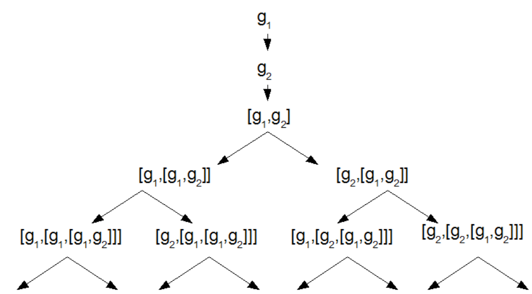

Figure 1: Lie Bracket Tree

This is only for a system given by equation 2 with two inputs, but it should make the concept more clear. If you have an n dimensional vector field, the Lie Bracket tree needs span R^n. So, for the vector field method, you are building up a set of linearly independent vector fields in order to check for controllability. Note: it is only necessary to determine one of the Lie Brackets at any given level, because the same result will be achieved for all Lie Brackets at the same level.

Introduction to Controllability Using Forms:

Before the concept of controllability using forms can be introduced, it is necessary to understand the notion of congruence. Two forms, γ and ξ, are said to be congruent mod I ($\gamma \equiv \xi \bmod I$) if:

$$\boldsymbol{\gamma = \xi + \eta \wedge \alpha} \qquad \textbf{(4)}$$

For $\alpha \in I, \eta \in \Omega$, where I is some set of forms on the manifold, and Ω is the set of all forms on the manifold.

Next, the concept of the derived flag needs to be introduced. The following is a very formal description of the derived flag, and it may be a bit confusing (admittedly, I am a little shaky on the formulation of the derived flag, but hopefully the example will make it clear how to use it in practice). Let $I = \text{span}\{\lambda^1, \ldots, \lambda^s\}$ be a smooth codistribution on M. The exterior derivative induces a mapping $\delta : I \rightarrow \Omega^2(M)/I$:

$$\boldsymbol{\delta : \lambda \rightarrow d\lambda \bmod I \in \Omega^2(M)} \qquad \textbf{(5)}$$

The mapping δ is a linear mapping over $C^{\infty}(M)$:

$$\begin{aligned} \boldsymbol{\delta(f\alpha + g\beta)} &= \boldsymbol{df \wedge \alpha + fd\alpha + dg \wedge \beta + fd\beta \bmod I} \\ &= \boldsymbol{fd\alpha + gd\beta \bmod I} \\ &= \boldsymbol{f\delta(\alpha) + g\delta(\beta)} \end{aligned}$$

The kernel of δ is a codistribution on M (i.e., at each point *p* on M, the kernel of δ is a linear subspace of T^*M). This subspace is known as the *first derived system* of *I*, and is given by:

$$\boldsymbol{I^{(1)} = ker\,\delta = \{\lambda \in I : d\lambda \bmod I \equiv 0\}} \qquad \textbf{(6)}$$

This can be continued, and a nested sequence of condistributions can be generated:

$$I = I^{(0)} \supset I^{(1)} \supset \ldots \supset I^{(N)} \quad \textbf{(7)}$$

If this terminates for some finite *N*, i.e. $I^{(N)} = \{\}$, then $I^{(N)}$ is known as the *derived flag* of *I*, and *N* is known as the *derived length*. In summary, you start out with a set of one forms, $I^{(0)}$, find the first derived system, and then continue finding the next derived system until you get to $I^{(N)}$, which should be empty. With every derived system that is found, the dimension of $I^{(i)}$ should decrease, until finally it terminates after the N-th derived system. This result is the key to determining controllability using forms, as outlined by Chow's Theorem.

Chow's Theorem: Let $I = \{\alpha^1, \ldots, \alpha^s\}$, represent a set of constraints and assume that the derived flag of the system exists. Then, there exists a path *x(t)* between any two points satisfying $\alpha^i(x)\dot{x} = 0$ for all *i* if and only if there exists an *N* such that $I^{(N)} = \{\}$.

Basically, what this theorem is saying is that given a set of constraints, the system is controllable if the derived flag exists ($I^{(N)} = \{\}$). Hopefully, this will be made clearer in the example to follow.

Example – Penny Rolling on a plane:

Consider a penny rolling on a plane, as depicted in Figure 2. Let $q \in \mathbb{R}^4$, where *(x,y)* denotes the location of the penny in the plane, θ is the angle that the penny makes with a fixed line in the plane, and ψ is the angle of rotation of the penny. For simplicity in the calculations, take the radius of the penny to be equal to 1.

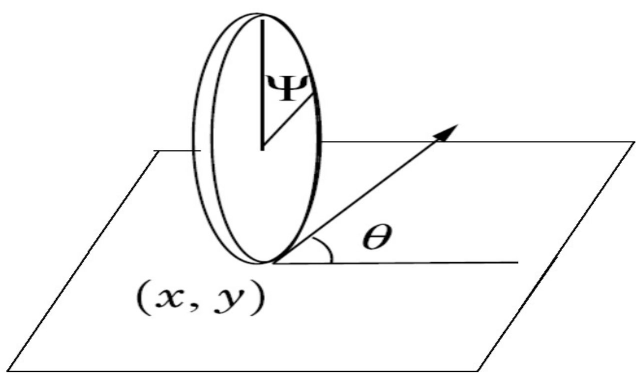

Figure 2: Penny Rolling on a Plane

In order to draw similarities between the vector field method and the forms method, in this example the steps of the two methods will be carried out together, instead of running through each method independently.

For the vector method, the system needs to be put into the drift-less control affine form given by equation 2. For this problem, the equations of motion can be written as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2 \quad \textbf{(8)}$$

For this example, u_1 is the input that rolls the penny, and u_2 is the input that changes the angle the penny makes with the fixed line on the plane.

For the forms method, n-2 constraints on the system are needed. For this problem, n = 4 so 2 constraints are needed.

$$I=\{\alpha_1,\alpha_2\} \tag{9}$$

The first constraint on the system is that all of the velocity is in the direction that the penny is rolling, and is equal to the angular velocity of the penny multiplied by its radius. Therefore:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}\cdot\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}=\dot{\psi}$$

$$\frac{dx}{dt}\cos(\theta)+\frac{dy}{dt}\sin(\theta)=\frac{d\psi}{dt}$$

$$dx\cos(\theta)+dy\sin(\theta)-d\psi=0$$

$$\alpha^1=\cos(\theta)dx+\sin(\theta)dy-d\psi \tag{10}$$

The next constraint is that the velocity must be zero in the direction perpendicular to the penny. This leads to:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}\cdot\begin{bmatrix}\sin(\theta)\\ -\cos(\theta)\end{bmatrix}=0$$

$$\frac{dx}{dt}\sin(\theta)-\frac{dy}{dt}\cos(\theta)=0$$

$$\sin(\theta)dx-\cos(\theta)dy=0$$

$$\alpha^2=\sin(\theta)dx-\cos(\theta)dy \tag{11}$$

So, to summarize the first steps, for the vector field method, you need to find the equations of motion of the system in the form of equation 2, and manipulate the following vectors:

$$g_1=\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\\ 0\\ 1\end{bmatrix},\quad g_2=\begin{bmatrix}0\\ 0\\ 1\\ 0\end{bmatrix}$$

For the forms method, you need to find n - 2 constraints (one forms) on the system and manipulate those. For this system, those constraints are given by equations 10 and 11.

The next step for the vector fields method is to determine the necessary Lie Brackets, which are:

$$[g_1, g_2] = \frac{\partial g_2}{\partial q^i} g_1 - \frac{\partial g_1}{\partial q^i} g_2 = \begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

$$[g_2, [g_1, g_2]] = \frac{\partial [g_1, g_2]}{\partial q^i} g_2 - \frac{\partial g_2}{\partial q^i} [g_1, g_2] = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

For the forms method, the exterior derivatives of equations 11 and 12 need to be found in order to determine the derived systems. For this system, the exterior derivatives are:

$$d\alpha^1 = -\sin(\theta)\, d\theta \wedge dx + \cos(\theta)\, d\theta \wedge dy \tag{14}$$

$$d\alpha^2 = \cos(\theta)\, d\theta \wedge dx + \sin(\theta)\, d\theta \wedge dy \tag{15}$$

Also, need to determine the exterior products in order to be able to determine which constraints pass through each level of the derived system. The exterior products of interest are:

$$d\alpha^1 \wedge \alpha^1 = -dx \wedge dy \wedge d\theta + \cos(\theta)\, dy \wedge d\theta \wedge d\psi - \sin(\theta)\, dx \wedge d\theta \wedge d\psi \tag{16}$$

$$d\alpha^1 \wedge \alpha^1 \wedge \alpha^2 = 0 \tag{17}$$

$$d\alpha^2 \wedge \alpha^1 \wedge \alpha^2 = -dx \wedge dy \wedge d\theta \wedge d\psi \neq 0 \tag{18}$$

Now it is time to check the controllability. For the vector field method, it is obvious that g_1 and g_2 are linearly independent, so starting out everything is okay.

For the forms method, $I^{(0)} = \{\alpha_1, \alpha_2\}$, so initially I is of dimension 2.

Next, looking at equation 12, it is clear that it is linearly independent of g_1 and g_2 , so the vector fields now span $\mathbb{R}^3$.

For the forms method, the first derived system is found to be $I^{(1)} = \{\alpha_1\}$. The reason for this is because equation 17 meets the requirements of equation 6, so α_1 carries through to the first derived system. However, because equation 18 is not equal to zero, α_2 does not move on to the first derived system. Therefore, the dimension is decreasing, so the system could be controllable.

Finally, for the vector fields method, it is clear that equation 13 is linearly independent of equation 12,g_1 and g_2 , so the vector fields span $\mathbb{R}^4$.

For the forms method, the next derived system is $I^{(2)}=\{\}$. This is because equation 16 is not equal to zero, so no constraint meets the requirements of equation 6. Therefore the derived flag exists, and the derived length is equal to 2.

Because the vector fields span $\mathbb{R}^4$, the system is found to be controllable using the vector fields method. For the forms method, because the derived flag was able to be found, the system is determined to be controllable using the forms method, so the two methods agree.

Conclusion:

The controllability of a system can be determined using either vector fields or forms; one method may be more desirable than the other depending on the problem. For the vector field method, one needs to have the equations of motion in the control affine form, and the input vector fields are manipulated to determine the controllability of the system. Also, when checking the controllability of the system, when moving down the Lie Bracket tree, you are checking to see if the dimension the vectors span is increasing. For the forms method, it is necessary to find a set of constraints on the system, and those constraints need to be manipulated to check the controllability of the system. When checking the controllability of the system, each consecutive derived system needs to be decreasing in dimension.

References:

1. R.M. Murray. Crash Course on Exterior Differential Systems. California Institute of Technology, May 1993.

2. S.D. Kelly. ME 598 – Geometric Mechanics. UIUC, Spring 2007.