An Introduction to Differential Flatness

William Burgoyne
ME598 – Geometric Mechanics
May 7, 2007

INTRODUCTION

Systems of ordinary differential equations (ODEs) can be divided into two classes, determined and underdetermined. The latter class can be defined as a system of m ODEs with n dependent variables where $n > m$. Systems of this sort are typically encountered by control engineers. They attempt to define $n - m$ of the dependent variables as functions of time in order force the system to act in a specified way. When the system of ODEs is linear, a number of methods exist to easily design control for the system. However, when the system of ODEs is nonlinear, control design is much more difficult and fewer design methods exist.

One method that has been developed to design control somewhat easily for nonlinear systems exploits a property of such systems known as differential flatness. A differentially flat system is a system of nonlinear ODEs whose solution curves are in a smooth one-one correspondence with arbitrary curves in a space whose dimension equals $n - m$. Hence generating solution trajectories for the system is simplified by finding curves in a lower dimension that satisfy conditions at the endpoints.

In this paper I provide an example of a differentially flat system and rigorously define differential flatness. I relate differential flatness to feedback linearization. I conclude with some general results of differential flatness and some remarks about finding flat outputs for Lagrangian systems.

MOTIVATIONAL EXAMPLE

Suppose it is desired to control the rigid body in the vertical plane as pictured in Figure 1. The set of ODEs defining this system is

$$\begin{aligned}
&m\ddot{x}_1 \sin\theta + m\ddot{x}_2 \cos\theta - mg\cos\theta = F_1 \\
&m\ddot{x}_1 \cos\theta - m\ddot{x}_2 \sin\theta + mg\sin\theta = -F_2 \\
&I\ddot{\theta} = -F_1 R
\end{aligned}$$

which is nonlinear. The system is underdetermined by two equations. The states of the system are x_1, x_2, θ and the inputs are F_1, F_2.

One method to control this system is to assign initial values to x_1, x_2, θ and their derivatives and assign F_1 and F_2 as arbitrary functions of time. The system becomes determined and evolves from a desired initial state to some final state. However, it would be difficult to determine the proper functions for F_1 and F_2 that result in the evolution of the system to a desired final value.

Another option to control the system is to simplify the problem by eliminating F_1 from the first and third ODEs. This results in a single ODE in terms of the states only

$$m\ddot{x}_1 \sin\theta + m\ddot{x}_2 \cos\theta - mg\cos\theta + (I / R)\ddot{\theta} = 0$$

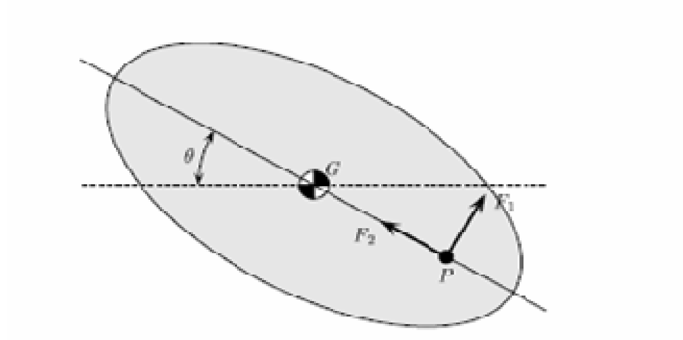

Figure 1 – Rigid body in the vertical plane.

which is also a nonlinear ODE. The system is still underdetermined by two equations. In this case, initial conditions need to be assigned for only one of the states and its derivative. The other two states are assigned as some desired functions of time. Once the solution for this ODE is obtained, then it can be used to determine F_1 and F_2. While this method ensures that two of the states evolve in a desired way, the third state may not. Again, it would be difficult to determine the proper functions for the two states that result in the evolution of the third state to a desired final value.

One might ask if there exists a combination of two dependent variables such that defining them as arbitrary functions of time would determine the system and cause the system to evolve as desired. The answer is no. However, one can form a pair of variables that are functions of the dependent variables and that determine the system without the need of defining initial conditions. One such combination is

$$y_1 = x_1 - (I/mR)\cos\theta$$
$$y_2 = x_2 + (I/mR)\sin\theta$$

where y_1 and y_2 are called flat outputs. These can be thought of as components of the map from the system space to the smaller dimension space. In other words, defining two new variables, which are functions of the dependent variables, as arbitrary functions of time determines the solutions for the system of five dependent variables. (Defining y_1 and y_2 so that the system meets control requirements is discussed in a following section.)

DEFINITION OF DIFFERENTIAL FLATNESS

A system is said to be *differentially flat* or simply *flat* if there exist variables $y_1,\dots,y_{n-m}$ given by an equation of the form

$$y = h(t, x, x', x'', \dots, x^k)$$

such that the original variables x may be recovered from y (locally) by an equation of the form

$$x = g(t, y, y', y'', \dots, y^l)$$

The variables $y_1,\dots,y_{n-m}$ are referred to as flat outputs. When y is only a function of time and the states the system is said to be *configuration flat*, which is the case for the preceding example.

TRAJECTORY GENERATION

Returning to the example, generating solutions that take the system from a desired initial condition to a desired final value is relatively easy. Given the initial conditions and final values for the states, their derivatives, and the inputs, the values for y_1, y_2, and their derivatives at the initial and final times can be determined. Then, one simply needs to generate functions of time that satisfy these initial and final conditions. It is interesting to note that connecting any starting and ending point in the system is possible. Hence, flat systems are controllable.

CONTROL SYSTEMS AND DIFFERENTIAL FLATNESS

When designing control for nonlinear systems, occasionally it is possible to transform the states in such a way that the control system can be expressed in a linear form. These systems are said to be feedback linearizable. It has been proven that a specific version of feedback linearizability is equivalent to differential flatness. Thus, there are potentially two ways to approach control design for certain nonlinear systems. One could use the feedback linearizability property of the system or to use the flatness property of the system.

RESULTS FROM DIFFERENTIAL FLATNESS

There do not exist general conditions that classify a system as flat or not. Also, there do not exist general conditions for determining the flat outputs of a differentially flat system. However, some specific properties do exist for flat systems. All controllable linear or state feedback linearizable systems are differentially flat. Differentially flat systems permit a transformation of the system such that the EOMs for the flat output variables become trivial. A performance index for a system can be approximated and transformed into a function of the flat outputs.

Lagrangian systems have been studied in detail for flatness and catalog exists detailing which types of Lagrangian systems are flat. The results of this study have revealed properties of these systems that indicate flatness. A common feature of rigid bodies with S^1 symmetry is that they evolve on manifolds which consist of, or include, a Lie group as part of the configuration space. Another common feature of rigid bodies with S^1 symmetry is that the flat outputs “break” symmetries because if the output is invariant with respect to some action, then the motion of the system can only be retrieved up to an initial choice of the group variable.

Another Lagrangian system that is flat is the system with three rigid bodies connected in a chain. This particular system and others are flat if certain parameters of the system are defined appropriately. A number of other mechanical systems are differentially flat and papers have been written explaining what the flat outputs are for those systems.

CONCLUSIONS

Nonlinear systems that have the property of differential flatness are relatively easy to control. Perhaps the most difficult part or using differential flatness is determining the flat outputs. Structural information contained in Lagrange's equations can be exploited to determine is a system is flat and how to find the flat outputs. While finding the flat outputs may be difficult, geometry can be used to aid in this task.

BIBLIOGRAPHY

- Murray, R. M., et al., "Differential flatness of mechanical control systems: A catalog of prototype systems" AMSE International Mechanical Engineering Congress and Expo, Nov 1995.
- Rathinam, M., "Differentially flat nonlinear control systems" Thesis, 1997.
- Rathinam, M. and Murray, R. M., "Configuration flatness of Lagrangian systems underactuated by one control" SIAM Journal on Control and Optimization, 36, 164, 1998.
- Rathinam, M. and Sluis, W. M., "A test for differential flatness by reduction to single input systems" Technical Memorandum No. CIT-CDS 95-018, June, 1995.
- Van Nieuwstadt, M., et al., "Differential flatness and absolute equivalence of nonlinear contol systems" SIAM Journal of Control and Optimization, 36, 1225, 1998.