An Introduction to Differential Flatness

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ME598 – Geometric Mechanics

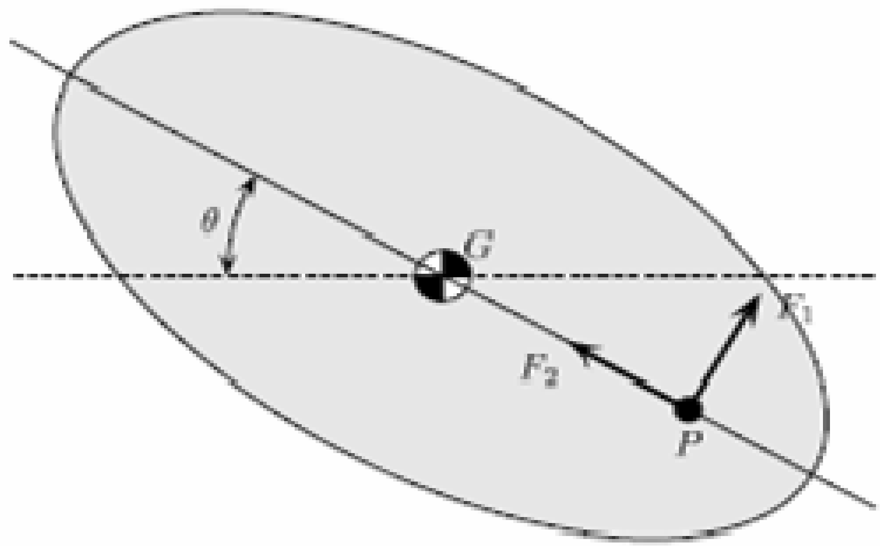
Definitions

- Underdetermined System – a system of m ODEs with n dependent variables where $m<n$
- Differentially Flat System – a system of nonlinear ODEs whose solution curves are in a smooth one-one correspondence with arbitrary curves in a space whose dimension equals $n-m$

Underdetermined Systems as Control Systems

- Dependent variables are separated into two groups: inputs and states
- Properties of underdetermined systems discovered through control system studies
  - Controllability
  - Linearization
  - Differential flatness
    - Generate solution trajectories by translating the problem to finding curves in a lower dimension that satisfy conditions at the end points

Example – Planar Rigid Body



$$m\ddot{x}_1 \sin\theta + m\ddot{x}_2 \cos\theta - mg\cos\theta = F_1$$

$$m\ddot{x}_1 \cos\theta - m\ddot{x}_2 \sin\theta + mg\sin\theta = -F_2$$

$$I\ddot{\theta} = -F_1 R$$

Example – Planar Rigid Body

- System is underdetermined by two equations and so has two control inputs
- System becomes determined by defining two of the variables as functions of time and defining some constants
  - Define F_1 and F_2 as functions of time
  - Assign initial values to x_1 , x_2 , θ , and their derivatives
  - This is not the unique way to determine the system

Example – Planar Rigid Body

- Simplify the analysis by eliminating F_1

$$m\ddot{x}_1 \sin\theta + m\ddot{x}_2 \cos\theta - mg\cos\theta + (I/R)\ddot{\theta} = 0$$

- System becomes determined by defining two of the variables as functions of time and defining two constants
- Set of solutions to this ODE is in one-one correspondence with the set of solutions to the previous set of ODEs

Example – Planar Rigid Body

- No combination of two dependent variables results in a determined system
- However, a pair of variables that are functions of the dependent variables can determine the system without the need of defining constants

$$y_1 = x_1 - (I / mR)\cos\theta$$
$$y_2 = x_2 + (I / mR)\sin\theta$$

- These are ‘flat outputs’

Rigorous Definition

A system is said to be *differentially flat* or simply *flat* if there exist variables $y^1,\ldots,y^p$ given by an equation of the form

$$y = h(t, x, x^{(1)},\ldots, x^{(m)})$$

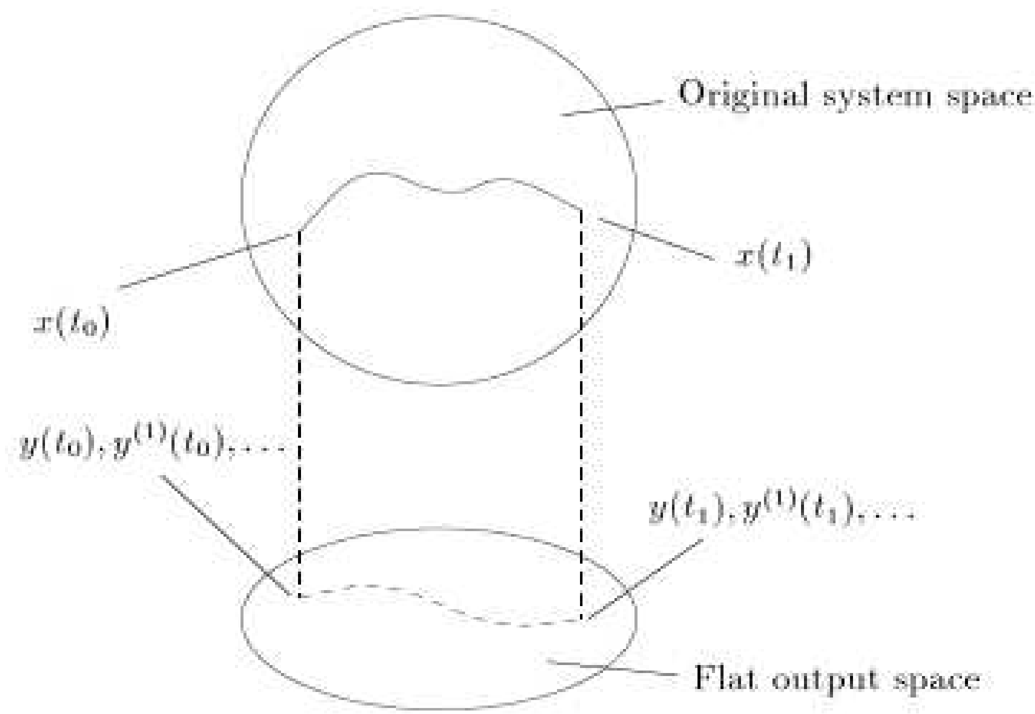
such that the original variables x may be recovered from y (locally) by an equation of the form

$$x = h(t, y, y^{(1)},\ldots, y^{(l)})$$

The variables $y^1,\ldots,y^p$ are referred to as *flat outputs*

Trajectory Generation

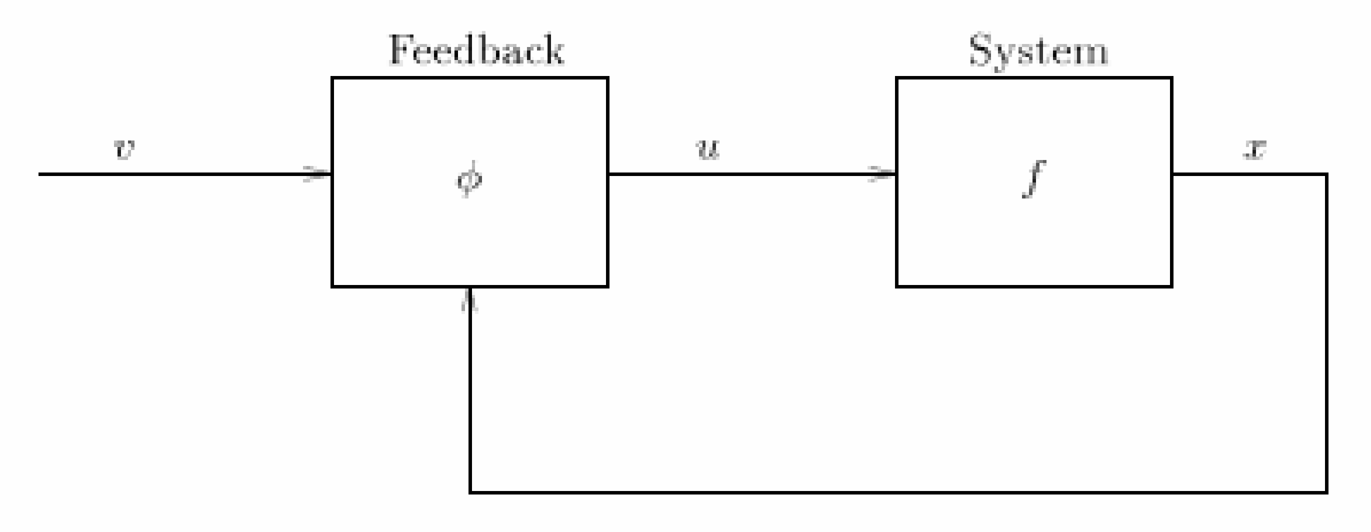
- Find solutions to a differentially flat system by translating the problem to the lower dimensional flat output space



Example – Planar Rigid Body

- Want to find a trajectory from some initial positions, velocities, and forces to some final positions, velocities, and forces
- All solutions to this problem can be mapped to curves on the flat output space with coordinates (y_1, y_2)
  - There is a local diffeomorphism between the variables y_1 , y_2 , and their derivatives to the system positions, velocities, and forces

Control Systems and Differential Flatness



- Some control systems are *feedback linearizable* and can be expressed in the linear form $\dot{\xi} = A\xi + Bv$ where $\xi = F(x, z)$
- Feedback linearizability is equivalent to differential flatness

Results of Differential Flatness

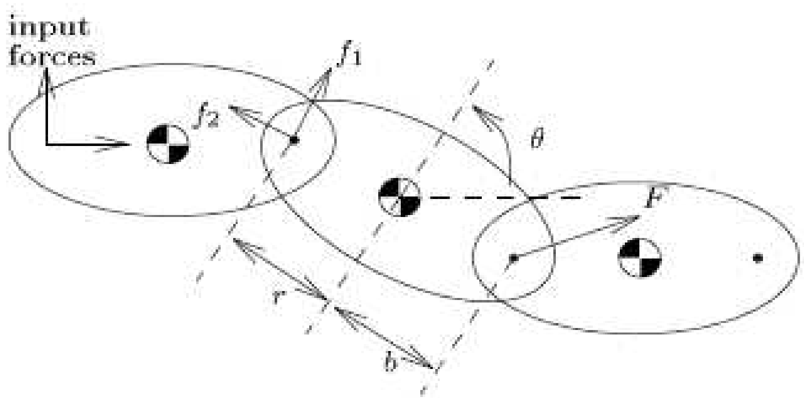
- All controllable linear or state feedback linearizable systems are differentially flat
- General conditions for differential flatness are not known
- Differentially flat systems permit a transformation of the system such that the EOMs for the flat output variables becomes trivial
- A performance index can be approximated and transformed into a function of the flat ouputs

Lagrangian Systems

- Rigid body with S^1 symmetry
  - Common feature is that they evolve on manifolds which consist of, or include, a Lie group as part of the configuration space
  - Flat outputs "break" symmetries because if the output is invariant with respect to some action, then the motion of the system can only be retrieved up to an initial choice of the group variable

Lagrangian Systems

- Coupled rigid bodies
  - Properly relate certain parameters of the system in order to determine flatness



- Other mechanical systems

Conclusions

- Differentially flat systems are useful when explicit trajectory generation is required; the problem simplifies to algebra
- Structural information contained in Lagrange's equations can be exploited
- Finding flat outputs is difficult, but is made easier using geometry

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