1995 ASME Int'l Mech Eng Congress and Expo
San Francisco, CA November 12-17, 1995

Differential Flatness of Mechanical Control Systems: A Catalog of Prototype Systems*

Richard M. Murray Muruhan Rathinam Willem Sluis

Division of Engineering and Applied Science
California Institute of Technology
Pasadena, CA 91125

Abstract

This paper describes the application of differential flatness techniques from nonlinear control theory to mechanical (Lagrangian) systems. Systems which are differentially flat have several useful properties which can be exploited to generate effective control strategies for nonlinear systems. For the special case of mechanical control systems, much more geometric information is present and the purpose of this paper is to explore the implications and features of that class of systems. We concentrate on several worked examples which illustrate the general theory and present a detailed catalog of known examples of differentially flat mechanical systems.

Keywords: nonlinear control, mechanical systems, trajectory generation, differential flatness.

1 Introduction

An emerging paradigm in nonlinear control is the use of two degree of freedom design techniques to generate nonlinear controllers for mechanical systems performing motion control tasks. The basic approach of two degree of freedom design is to initially separate the nonlinear controller synthesis problem into design of a feasible trajectory for the nominal model of the system, followed by regulation around that trajectory using controllers that have guaranteed performance in the presence of uncertainties. This splitting of the problem offers several advantages over existing nonlinear methods and allows the use of advances in linear controller synthesis to help achieve robust performance. Sample applications include high-performance control of piloted aircraft using vectored thrust propulsion, navigation and control of unmanned flight vehicles performing surveillance and other tasks, motion control and stabilization of underwater vehicles and ships, and control of land-based robotic locomotion systems.

One of the classes of systems for which trajectory generation is particularly easy are so-called differentially flat systems. Roughly speaking, a system is differentially flat if we can find a set of outputs (equal in number to the number of inputs) such that all states and inputs can be determined from these outputs without integration. More precisely, if the system has states $x \in \mathbb{R}^n$, and inputs $u \in \mathbb{R}^m$ then the system is flat if we can find outputs $y \in \mathbb{R}^m$ of the form

$$y = y(x, u, \dot{u}, \dots, u^{(p)}) \tag{1}$$

such that

$$\begin{aligned} x &= x(y, \dot{y}, \dots, y^{(q)}) \\ u &= u(y, \dot{y}, \dots, y^{(q)}). \end{aligned} \tag{2}$$

Differentially flat systems are useful in situations where explicit trajectory generation is required. Since the behavior of flat system is determined by the flat outputs, we can plan trajectories in output space, and then map these to appropriate inputs.

Differentially flat systems were originally studied by Fliess et al. in the context of differential algebra [4] and later using Lie-Bäcklund transformations [5]. In [27] we reinterpreted flatness in a differential geometric setting. We made extensive use of the tools offered by exterior differential systems and the ideas of Cartan. Using this framework we were able to recover most of the results currently available using the differential algebraic formulation and achieve a more geometric understanding of flatness.

In this paper we concentrate on characterizing differential flatness for *mechanical* control systems. More specifically, we attempt to exploit the structure of second order systems whose unforced motion is described by Lagrangian mechanics. There are several existing results which indicate that differential flatness for this class of systems is highly dependent on the special structure available due to the Lagrangian nature of the system. For example, it can be shown that all of the following systems are differentially flat: a car pulling N trailers with the hitch of the ith trailer attached at the axle of the preceding vehicle in the chain [22, 24, 25]; any planar rigid body with forces whose lines of action do not intersect at the center of mass [14]; an airplane towing a cable with a rigid body attached at the end of the cable [19]; and a satellite system with three control torques and a single thruster whose line of action intersects the center of mass.

In all of these examples, the differentially flat output is not an arbitrary combination of the configuration variables and velocities of the system, but rather consists of a set of points and angles. The exact reason for this is not yet understood,

*This work supported in part by NSF grant CMS-9502224 and AFOSR grant F49620-95-1-0419.

but certainly involves the second order nature of the system, combined with symmetry relations (when present) and the structure of the inertia tensor for the system.

The implications of flatness for these systems is that the trajectory generation problem can be reduced to simple algebra, in theory, and computationally attractive algorithms in practice. For example, in the case of the towed cable system, a reasonable state space representation of the system consists of approximately 128 states. Traditional approaches to trajectory generation, such as optimal control, cannot be easily applied in this case. However, it follows from the fact that the system is differentially flat that the feasible trajectories of the system are completely characterized by the motion of the point at the bottom of the cable. By converting the input constraints on the system to constraints on the curvature and higher derivatives of the motion of the bottom of the cable, it is possible to compute efficient techniques for trajectory generation.

In this paper we present initial results on the characterization of differential flatness for mechanical systems and show how symmetries and inertial properties relate to differential flatness. We give a complete characterization of differential flatness for a planar rigid body with one, two or three body-fixed forces and present more specialized results for rigid bodies in $\mathbb{R}^3$ and coupled rigid bodies.

2 Differential Flatness

Differential flatness was originally characterized by Fliess et al. [4] using tools from differential algebra. While this technique is quite powerful, it is difficult to apply differential algebraic results to systems with strong geometric structure while at the same time exploiting that structure. In [27] we gave an alternative characterization of flatness using using tools from differential geometry, and in particular within the framework of exterior differential systems. In this section we briefly review those results and attempt to cast them into a somewhat less formal framework for the benefit of readers who are not familiar with the details of exterior differential systems.

2.1 Basic definitions

Formally, we work within the context of Pfaffian systems (see [2] for a detailed description). For the purposes of this paper, it is sufficient to work in local coordinates and we assume all functions and maps are smooth (C^∞). For a more rigorous treatment of the specific material summarized here, see [27].

Roughly speaking, we convert a control system of the form

$$\dot{x} = f(x,u) \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

into a Pfaffian system by "multiplying through by dt" and writing the system as a collection of forms,

$$I = \{dx_1 - f_1(x,u)dt, \ldots, dx_n - f_n(x,u)dt\}. \qquad (3)$$

The collection I is called a *Pfaffian system* on $M = \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m$, where $\mathbb{R}_+$ is the time coordinate t. It is important to note that while we often study a nonlinear control system in terms of vector fields on $\mathbb{R}^n$, the one forms in a Pfaffian system are objects on a larger space. We will write (I, M) when necessary to indicate both the Pfaffian system and the space on which it lives.

A *solution* or *integral curve* of a Pfaffian system is a curve $c : \mathbb{R} \to M$ such that the tangent vector $c'(s)$ is annihilated by the one forms in the Pfaffian system I. We usually also impose an *independence condition*, which is an extra one-form τ on which $c'(s)$ is required not to vanish. A standard choice for control systems is to choose $\tau = dt$, which insures that time is always increasing. We write (I, τ) or $(I, \tau; M)$ for a Pfaffian system I together with independence condition τ (on a manifold M).

Pfaffian systems can be used to study controllability and linearizability properties of nonlinear control systems in much the same way that vector fields are used. The exterior derivative plays the role of Lie brackets and the derived flag plays the role of the controllability distribution. We will omit a complete discussion, concentrating instead of the results directly applicable to differential flatness. It is interesting to note that many of the main techniques in exterior differential systems have been in place since the 1920s while the corresponding vector field versions of those results became available only in the 1970s. Indeed, the original results by Chow which are usually the starting point for modern geometric nonlinear control theory were actually done in the context of differential forms [3].

An essential operation on Pfaffian systems is that of prolongation. Given a Pfaffian system $(I; M)$ and another Pfaffian system $(J; N)$ with $M \subset N$, we say that J is a *Cartan prolongation* of I if $I \subset J$ and there is a one-to-one correspondence of solutions curves on J with solution curves on I. A special case of Cartan prolongation is a *prolongation by differentiation*, in which one of the inputs is differentiated with respect to time (for the special case of a control system). For example, we can have

$$\begin{aligned} I &= \{dx - f(x,u)dt\} \\ M &= \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \\ J &= \{dx - f(x,u)dt, du_1 - v_1 dt\} \\ N &= M \times \mathbb{R} \end{aligned}$$

corresponding to dynamic extension of the first input. If *all* of the inputs of a control system are differentiated with respect to time, the extended system is a *total prolongation* of the original control system. The coordinate v_1 in the previous example is called a *fiber coordinate* for the mapping $\pi : N \to M$ given by projection.

Although the use of dynamic extension is common in nonlinear control theory, it is important to note that Cartan prolongations are more general than dynamic extension. This distinction is important, for example, when dealing with mechanical systems with nonholonomic constraints.

Cartan prolongations allow us to define a notion of equivalence which allows, among other things, equivalence of control systems under certain types of dynamic feedback. We say that two Pfaffian systems (I_1, M_1) and (I_2, M_2) are *absolutely equivalent* if there exist respective Cartan prolongations (J_1, N_1) and (J_2, N_2) such that J_1 and J_2 are equivalent in the ordinary sense, i.e., there exists a diffeomorphisms $\phi : N_1 \to N_2$ such that the pullback of J_2 is equal to J_1: $\phi^* J_2 = J_1$. The following diagram captures this definition:

$$\begin{array}{ccc} J_1 & \overset{\phi}{\longleftrightarrow} & J_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ I_1 & & I_2 \end{array}$$

With these definitions in place, we are now in a position to state the formal definition for differential flatness. We make use of the notion of a *trivial system* on a manifold M, which corresponds to a zero Pfaffian system (no one-forms). For such a system, any curve on M is an integral curve. We write $(\{0\}, dt)$ for the trivial system with independence condition $\tau = dt$.

Definition 1 ([27]). A Pfaffian system (I, dt) is *differentially flat* if it is absolutely equivalent to the trivial system $(\{0\}, dt)$.

Notice that we require that the independence condition be preserved by the Cartan prolongations and diffeomorphisms, and hence our notion of time is the same for both systems. However, we do allow time to enter into the various mappings from one system to the other. If time is not required in any of the various mappings then we say the system is *time-independent differentially flat.*

The following lemma establishes the relationship between our definition and the differential algebraic notion of flatness.

Lemma 1 ([27]). *Let (I, dt) be a system on a manifold M with local coordinates $(t, x) \in \mathbb{R}^1 \times \mathbb{R}^n$ and let (J, dt) be a Cartan prolongation on the manifold $\pi : N \to M$ with fiber coordinates $y \in \mathbb{R}^r$. Then, under suitable regularity conditions and on an open dense set, each y_i can be uniquely determined from t, x and a finite number of derivatives of x.*

Example 1. As a simple example, consider a chain of integrators,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= u. \end{aligned}$$

This system is absolutely equivalent to the system $(0, dt)$ on $\mathbb{R}_+ \times \mathbb{R}$ given by $(t, x) \mapsto (t, x_1)$. To see, this consider the correspondence of solution curves of the chain of integrators with curves in x_1: Given a solution for the full control system we clearly get a well-defined curve for x_1 and, conversely, given any $x_1(t)$ we can find a unique solution of the full control system by differentiation of x_1.

2.2 Known results

As the previous example shows, chains of integrators, and in fact all controllable linear or state feedback linearizable systems, are differentially flat. The converse is also true on an open dense set if we allow dynamic feedback (although equilibrium points may not be included in the dense set, so this statement should be taken with care). In this section we summarize some of the various results which exist for differentially flat systems.

We recall that a control system

$$\dot{x} = f(x, u) \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

is *full-state feedback linearizable* if there exists a change of coordinates $\xi = \phi(x)$ and a feedback control law $u = \alpha(x) + \beta(x)v$ with β full rank, such that

$$\dot{\xi} = A\xi + Bv.$$

A system is *dynamic feedback linearizable* if there is a dynamic compensator of the form

$$\begin{aligned} \dot{z} &= a(x, z) + b(x, z)v \\ u &= c(x, z) + d(x, z)v \end{aligned} \qquad z \in \mathbb{R}^p, v \in \mathbb{R}^m$$

such that the original system plus dynamic feedback is full state linearizable.

Every system which is dynamic feedback linearizable via an "endogenous" feedback (roughly, an invertible dynamic feedback; see [11, 12] for details) is differentially flat by using state feedback to convert the system to a chain of integrators and then choosing the outputs at the end of each integrator chain as flat output. The converse is also true, on an open and dense set:

Theorem 2 ([27, 15]). *If a control system is differentially flat then it is dynamic feedback linearizable on an open dense set, with the dynamic feedback possibly depending explicitly on time.*

Much more can be said in the case of single input systems. In particular, it can be shown that explicit time dependence and dynamic feedback are both unnecessary in the single input case. These results stem in part from that fact that in codimension two (i.e., the number of one-forms in I is two less than the dimension of M) it can be shown that every Cartan prolongation is a total prolongation and that total prolongations do not affect feedback linearizability (see [23]). We state these results as two theorems:

Theorem 3. *Let I be a differentially flat, autonomous control system (with a possibly time varying flat output),*

$$I = \{dx_1 - f_1(x, u)dt, \ldots, dx_n - f_n(x, u)dt\},$$

where u is a scalar control, i.e., the system has codimension two. Then I is feedback linearizable by static autonomous feedback.

Theorem 4. *Let I be an single input autonomous control system,*

$$I = \{dx_1 - f_1(x,u)dt, \ldots, dx_n - f_n(x,u)dt\}.$$

If I is time-independent differentially flat around an equilibrium point, then I is feedback linearizable by static autonomous feedback at that equilibrium point.

Most of the other results which are known for differential flatness of are in the context of low-dimensional examples, where it is possible to directly work out the various cases in detail to check for flatness. Many of these results are summarized in the following theorem:

Theorem 5 ([11, 16]). *Every controllable, codimension three Pfaffian system with no more than five states is differentially flat.*

Unfortunately, general conditions for flatness are not known, but all (dynamic) feedback linearizable systems are differentially flat, as are all driftless systems which can be converted into chained form (see [27] for details). Another large class of differentially flat systems are those in "pure feedback form":

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2, x_3)\\ &\vdots\\ \dot{x}_n &= f_n(x_1, \ldots, x_n, u).\end{aligned}$$

Under certain regularity conditions these systems are differentially flat with output $y = x_1$. These systems have been used for so-called "integrator backstepping" approaches to nonlinear control by Kokotovic et al. [9].

2.3 The utility of differentially flat systems

The advantage of using differentially flat outputs, when they are available, is similar to that of using configuration space approaches to path planning for robot manipulators (see [10] for a detailed discussion of these methods). In configuration space based motion planning, one converts the geometry of the problem for the workspace (usually $SE(3)$) to the configuration space of the manipulator. This has the effect of reducing the robot to a point which can then be guided through the transformed obstacles using any number of generic algorithms.

Similarly, for differentially flat systems we are able to transform the system such that the equations of motion for the flat output variables become trivial. Since the flat output functions are completely free, the only constraints that must be satisfied are the initial and final conditions on the endpoints, their tangents, and higher order derivatives. Any other constraints on the system, such as bounds on the inputs, can be transformed into the flat output space and (typically) become limits on the curvature or higher order derivative properties of the curve.

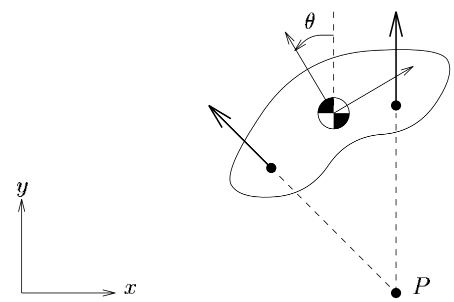


Figure 1: A rigid body controlled by two body fixed forces.

If there is a performance index for the system, this index can be transformed and becomes a functional depending on the flat outputs and their derivatives up to some order. By approximating the performance index we can achieve paths for the system which are suboptimal but still feasible. This approach is often much more appealing than the traditional method of approximating the system (for example by its linearization) and then using the exact performance index, which yields optimal paths but for the wrong system.

Some initial computational approaches along the lines discussed here can be found in [26].

3 Differential Flatness of Lagrangian Systems

Differential flatness is a concept which was originally defined in the context of general first order control systems evolving on a manifold. In this section, we concentrate on Lagrangian systems and indicate how the special structure of mechanical systems can be used to determine if a system is differentially flat.

3.1 Motivating example: planar rigid body

Consider a planar rigid body moving in a vertical plane under the influence of gravity and controlled by two forces having lines of action that are fixed with respect to the body and intersect at a single point (see Figure 1). Let (x, y) represent the horizontal and vertical coordinates of center of mass G of the body with respect to a stationary frame, and let θ be the counterclockwise orientation of a body fixed line through the center of mass. The choice of this line will be made to simplify algebra depending on the case being considered. Take m as the mass of the body and J as the moment of inertia. Let $g \approx 9.8$ m/sec^2 represent the acceleration due to gravity.

Without loss of generality, we will assume that the lines of action for F_1 and F_2 intersect the y axis of the rigid body and that F_1 and F_2 are perpendicular. The equations of

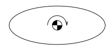
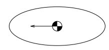
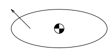
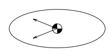
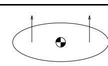
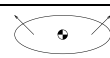
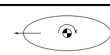
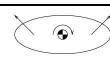
Inputs	Picture	Controllable?	Flat Output
Single torque		no	not flat
Single force, at center of mass		no	not flat
Single force, off center		yes	not flat
Two forces, at center of mass		no	not flat
Two colinear forces, off center		yes	center of mass
Two noncolinear forces, off center		yes	center of oscillation
One force, one torque		yes	center of mass
Two noncolinear forces, one torque		yes	center of mass

Table 1: Flatness of a rigid body with body fixed forces and torques. For multiple inputs, the forces/torques are assumed to generate independent generalized forces in $SE(2)$. In addition, all forces and torques are bidirectional.

motion for the system can be written as

$$\begin{aligned} m\ddot{x} &= F_1 \cos\theta - F_2 \sin\theta \\ m\ddot{y} &= F_1 \sin\theta + F_2 \cos\theta - mg \\ J\ddot{\theta} &= rF_1, \end{aligned} \tag{4}$$

which we rewrite as

$$\begin{aligned} F_1 - mg\sin\theta &= m\ddot{x}\cos\theta + m\ddot{y}\sin\theta \\ F_2 - mg\cos\theta &= -m\ddot{x}\sin\theta + m\ddot{y}\cos\theta \\ F_1 r &= J\ddot{\theta}. \end{aligned} \tag{5}$$

F_1 can be eliminated from the first and third equations to yield

$$-\frac{J}{mr}\ddot{\theta} + \ddot{x}\cos\theta + \ddot{y}\sin\theta + g\sin\theta = 0. \tag{6}$$

Martin et al. [14] showed that this system is flat and that the flat outputs z_1 and z_2 are given by

$$\begin{aligned} z_1 &= x - \frac{J}{mr}\sin\theta \\ z_2 &= y + \frac{J}{mr}\cos\theta. \end{aligned} \tag{7}$$

Substituting $x = z_1 + \frac{J}{mr}\sin\theta$ and $y = z_2 - \frac{J}{mr}\cos\theta$ into equation (6) we obtain

$$\ddot{z}_1 \cos\theta + (\ddot{z}_2 + g)\sin\theta = 0. \tag{8}$$

This shows that given $z_1(t)$ and $z_2(t)$ we can find $\theta(t)$ except for an ambiguity of π and away from the singularity $\ddot{z_1} = \ddot{z_2} + g = 0$. The forces $F_1(t)$ and $F_2(t)$ can then be obtained from the equations of motion.

It is interesting to observe that the flat outputs correspond to the coordinates of a body fixed point, a point that is on the line joining the center of mass G and point of intersection P of the forces. This point is distance $\frac{J}{mr}$ from G on the other side of P. This point is historically known as *center of oscillation* and becomes important in the study of planar rigid pendulums. When this planar rigid body is fixed at P and allowed to oscillate as a pendulum, its equations of motion will be identical to that of a point mass pendulum obtained by concentrating the mass m at the center of oscillation distance $r + \frac{J}{mr}$ from the pivot P.

This example has some practical importance as well. The PVTOL system studied by Hauser et al. [8] is exactly of this form, as is the simplified *planar ducted fan* described in [26]. Variations of this example can be formed by changing the number and type of the inputs. Table 1 summarizes the different possible choices of inputs and indicates which ones are controllable and which ones are flat.

3.2 Symmetries and flatness: the rigid body with $\mathbf{S^1}$ symmetry

A common feature of many examples of differentially flat mechanical systems is that they evolve on manifolds which consist of, or include, a Lie group as part of the configuration space. For example, the ducted fan evolves on $SE(2)$, the group of Euclidean motions in the plane. Problems involving mobile robots also naturally include this group as part of the the configuration space. Flight vehicle systems are described by their internal shape and their position and attitude, the latter two quantities taking values in $SE(3)$, the group of Euclidean motions on $\mathbb{R}^3$

In addition, the uncontrolled dynamics for many of the known examples for flat systems have symmetries with respect to an action of the Lie group on the configuration space of the system. For unconstrained systems, these symmetries lead to conservation laws (for the constrained case, see [1]).

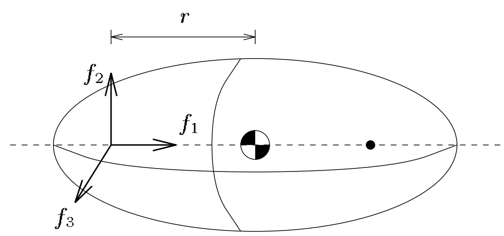


Figure 2: A rigid body in $\mathbb{R}^3$ with S^1 symmetry.

It is clear that the flat outputs for the system must "break" symmetries, since if the flat output is invariant with respect to some action, then the free motion of the system can only be retrieved up to an initial choice of the group variable. A more complete understanding of symmetry breaking in this context remains to be established but seems to be a common feature of many problems.

To illustrate how symmetry and reduction can affect flatness, we consider the example of a rigid body in $\mathbb{R}^3$ with three body fixed forces all acting at a single point. This example was originally presented in [26] and is the generalization of the planar rigid body example to $\mathbb{R}^3$. Since there are three principle moments of inertia, the center of oscillation is no longer a well defined quantity and detailed calculations have failed to find a body fixed point which is a flat output.

Suppose now that the rigid body is actually a surface of revolution about an axis which passes through the application point of the three forces, as shown in Figure 2. This is roughly the configuration of a submarine (in a vacuum), where the body consists of an ellipsoid with forces applied by the propeller and vectored using a set of rudders. The equations of motion for such a system are given by

$$\begin{bmatrix} m\ddot{x} \\ m\ddot{y} \\ m\ddot{z} \end{bmatrix} = RF + \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}$$
$$\begin{bmatrix} J_1\dot{\omega}_1 \\ J_2\dot{\omega}_2 \\ J_3\dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ (J_3 - J_1)(J_1^{-1}\mu)\omega_3 + rF_z \\ (J_1 - J_2)(J_1^{-1}\mu)\omega_2 - rF_y \end{bmatrix} \tag{9}$$

where $(x, y, z) \in \mathbb{R}^3$ is the position of the center of mass, $R \in SO(3)$ is the orientation, $F \in \mathbb{R}^3$ is the vector of body forces, $m \in \mathbb{R}$ is the mass, $J_1, J_2, J_3 \in \mathbb{R}$ are the principal moments of inertia, $\omega \in \mathbb{R}^3$ is the body angular velocity, and $\mu := J_1\omega_1$ is the conserved body angular momentum about the axis of revolution. Note that $J_2 = J_3$ due to the body symmetry.

We choose as our candidate flat output the function corresponding to the center of oscillation:

$$w = p + R \begin{bmatrix} \frac{\bar{J}}{mr} \\ 0 \\ 0 \end{bmatrix}, \tag{10}$$

where $\bar{J} = (J_2 + J_3)/2$. A detailed but otherwise straightforward computation shows that

$$m\ddot{w} = R \begin{bmatrix} F_x - (\bar{J}/r)\|\omega\|^2 - (\bar{J}/r)\omega_1^2 \\ (\mu/r)\omega_2 \\ (\mu/r)\omega_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}.$$

If $\mu = 0$ then given the flat outputs $w(\cdot)$ we can determine the direction of the first column of the rotation matrix, which contains all of the information about the orientation up to rotation about the first axis. However, if $\mu = 0$ then this angle is fixed and hence we can recover the entire configuration of the rigid body. Further differentiations lead to expressions for the velocity and forces as a function of the motion of the flat output.

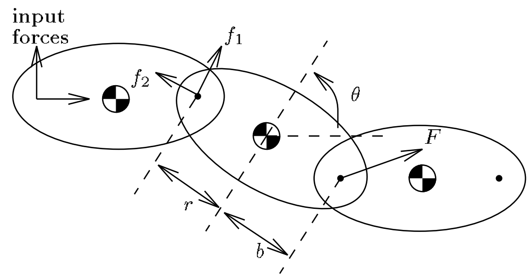


Figure 3: A chain of coupled rigid bodies in the plane.

3.3 Flatness of coupled rigid bodies

In some cases, a mechanical system may be differentially flat only for certain choices of parameters. The previous example required that two of the principal moments of inertia to be equal. In this section, we consider a somewhat different example where we don't rely explicitly on body symmetries, but we do require that certain parameters of the system be properly related in order to determine flatness.

Consider the case of a chain of coupled rigid bodies, with a pair of forces acting on the first body in the chain, as shown in Figure 3. We assume that the line connecting the interconnection points on each rigid body goes through the center of mass and that the forces applied to the first body are on the line formed by the interconnection point and the center of mass. From the example in Section 3.1, it is clear that if we choose the center of oscillation of the last rigid body in the chain as an output, then we can determine the motion of that rigid body as well as the forces applied at the interconnection point.

Consider now the next rigid body in the chain (moving from right to left in the figure). The equations of motion are given by

$$\begin{aligned} m\ddot{x} &= f_1 \cos\theta - f_2 \sin\theta + F_1 \\ m\ddot{y} &= f_1 \sin\theta + f_2 \cos\theta - mg + F_2 \\ J\ddot{\theta} &= -rf_1 + b(F_1 \cos\theta + F_2 \sin\theta), \end{aligned}$$

where (f_1, f_2) is the pair of forces from the next rigid body in the chain, written in body coordinates; r is the distance from the center of mass to the interconnection point of the next rigid body; (F_1, F_2) is the force from the last rigid body (whose motion is known), written in spatial coordinates; b is the distance from the center of mass to the interconnection point of the last rigid body; and the rest of the variables are as in the single rigid body case.

We know the motion of the point of attachment between the current rigid body and the last rigid body as well as the forces F_1 and F_2. Our goal is to determine the motion of the attachment point of the next rigid body, as well as the forces f_1 and f_2. Let (z_1, z_2) be the coordinates of the point of attachment,

$$z_1 = x + b\sin\theta$$
$$z_2 = y - b\cos\theta.$$

Computing the second derivative of z along the flow of the system yields

$$m\ddot{z}_1 = f_1 c_\theta - f_2 s_\theta + F_1 + mbc_\theta\big(-\frac{r}{J}f_1 + \frac{b}{J}F_1 c_\theta + \frac{b}{J}F_2 s_\theta\big) - mbs_\theta\dot{\theta}^2,$$

where $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$, and a similar equation is obtained for $m\ddot{z}_2$. If we choose $b = \frac{J}{mr}$ then the equations for $\ddot{z}_1$ and $\ddot{z}_2$ become

$$m\ddot{z}_1 = s_\theta(-f_2 + \frac{J}{mr^2}c_\theta F_2 - \frac{J}{r}\dot{\theta}^2) + F_1(1 + \frac{J}{mr^2}c_\theta^2)$$
$$m\ddot{z}_2 = c_\theta(f_2\frac{J}{mr^2}s_\theta F_1 + \frac{J}{r}\dot{\theta}^2) + F_2(1 + \frac{J}{mr^2}s_\theta^2) - mg.$$

Rewriting s_θ^2 and c_θ^2, we obtain

$$m\ddot{z}_1 - F_1(1 + \frac{J}{mr^2}) = - s_\theta(f_2 + \frac{J}{mr^2}s_\theta F_1 - \frac{J}{mr^2}c_\theta F_2 + \frac{J}{r}\dot{\theta}^2)$$

$$m\ddot{z}_2 - F_2(1 + \frac{J}{mr^2}) + mg = c_\theta(f_2 + \frac{J}{mr^2}s_\theta F_1 - \frac{J}{mr^2}c_\theta F_2 + \frac{J}{r}\dot{\theta}^2)$$

The left hand side consists of known quantities and hence we can determine θ by computing the ratio of the two equations.

The motion of the remainder of the chain is determined by recursion, giving the motion of the entire chain and the forces applied to the first rigid body as a function of the trajectory of the center of oscillation of the last rigid body. Thus, the system is differentially flat using the position center of oscillation of the last body in the chain as the flat output and assuming $b = \frac{J}{mr}$.

A more general procedure for determining constraints between parameters of the system is described in [21]. In that paper, one fixes the flat outputs up to a set of undetermined constants or functions and then derives conditions on the outputs and the parameters of the system in order to make the system flat. It is shown, for example, that if we add torques to the joints in the two coupled rigid bodies example, then the system can be made flat by using a body fixed point on the outer rigid body combined with a non-obvious linear combination of the orientation angles of the two bodies. In that case, the rigid bodies need not be attached at the centers of oscillation and there are no other constraints on the parameters of the system.

3.4 Other mechanical examples

Over the past several years, a number of examples of mechanical systems which are differentially flat have emerged. Table 2 gives a partial list of these systems, along with references to more detailed information. Additional information is also available via the World Wide Web at URL `http://avalon.caltech.edu/~murray/mechsys.html`.

4 Summary

In this paper we have summarized some of the recent results in differential flatness of nonlinear control systems and indicated some of the specialized results that are possible by restricting attention to mechanical systems. This area of work is part of an ongoing research effort in nonlinear control of mechanical systems, with particular application to problems in robotics, flight control, and space vehicles. For all of these systems, the structural information contained in Lagrange's equations must be exploited to achieve maximum performance over a wide range of operating conditions.

The advantage of using differentially flat outputs when they are available is that the problem of trajectory generation reduces to algebra instead of a problem in dynamic programming. This is true even if the flat outputs are not the actual outputs which are to be tracked (see [26] for a discussion). Examples of differentially flat mechanical systems include planar rigid bodies with body fixed forces, mobile robots with and without trailers, and flight control systems such as the towed cable system.

Much work remains to be done on this important class of systems, both from the theoretical perspective and in the context of applications. At the present, constructive conditions for finding the flat outputs of a mechanical system are not available except in a few special (i.e. low-dimensional) cases. In addition, for systems which are not differentially flat, it is likely that approximations can be used which will allow fast and efficient generation of approximately feasible trajectories. Bounds on the sizes of the error in the performance of the system as a function of the degree of approximation will be needed in order to pursue efforts in this direction.

Acknowledgements

The authors would like to thank Michiel van Nieuwstadt for his help in preparing this paper.

References

[1] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, and R. M. Murray. Nonholonomic mechanical systems with

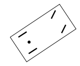
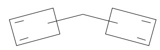
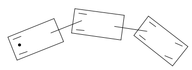
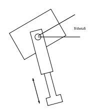
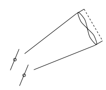
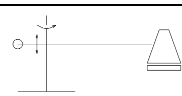
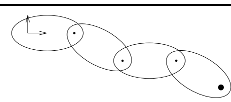
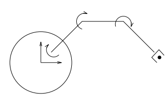
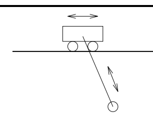
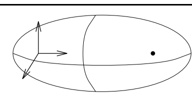
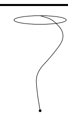
System	Picture	Flat Output	Comments	Reference
Mobile robot, car		rear wheel position	perfect rolling	See [20]
Car with 1 trailer		function of hitch angle/position	perfect rolling	[15]
Car with N trailers, special hitching		last trailer wheel position	perfect rolling	[22, 24, 25]
Hopping robot		position of end of leg		See [20]
Ducted fan/PVTOL		center of oscillation	no drag	[14], §3.1
Ducted fan with ideal stand		quasi-center of oscillation	$m_x \neq m_y$, no drag	[26]
Planar rigid body chain		center of oscillation, last rigid body	attachment points at centers of oscillation	[13], §3.3
Planar satellite with actuated robotic arm		body fixed point plus linear functions of joint angles		[21]
Simplified planar crane		position of the load		[6, 7]
Rigid body with S^1 symmetry and body fixed forces in $\mathbb{R}^3$		axisymmemtric center of oscillation		[26], §3.2
Towed cable system		position of end of cable	Drag doesn't destroy flatness	[19]
Fully actuated mechanical system	$M(q)\ddot{q}+N(q,\dot{q}) = \tau$	configuration coordinates	covers robot manipulators	See [18]
(Dynamic) feedback linearizable system	$\dot{x} = f(x,u)$	end of integrator chain	Equivalent to flatness on open dense set	See [4, 15, 27]
Nonholonomic system with 2–4 states, 2 controls	$\dot{x} = g(x)u$	system dependent	Assume controllable	[17, 12]
Nonholonomic system with 5 states, 3 controls	$\dot{x} = g(x)u$	system dependent	Assume controllable	[16]
Nonholonomic system in chained form with single generator	$\dot{x} = g(x)u$	first and last states		See [20]

Table 2: A partial catalog of differentially flat mechanical systems.

symmetry. Technical Report CIT/CDS 94-013, California Institute of Technology, 1994. Submitted to *Archive for Rational Mechanics*. Available electronically via `http://avalon.caltech.edu/cds/`.

[2] R. L Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths. *Exterior Differential Systems*. Springer-Verlag, 1991.

[3] W-L. Chow. Über systeme von linearen partiellen differentialgleichungen erster ordnung. *Math. Annalen*, 117:98–105, 1940.

[4] M. Fliess, J. Levine, P. Martin, and P. Rouchon. On differentially flat nonlinear systems. *Comptes Rendus des Séances de l'Académie des Sciences*, 315:619–624, 1992. Serie I.

[5] M. Fliess, J. Levine, P. Martin, and P. Rouchon. Linéarisation par bouclage dynamique et transformations de Lie-Bäcklund. *Comptes Rendus des Séances de l'Académie des Sciences*, 317(10):981–986, 1993. Serie I.

[6] M. Fliess, J. Lévine, and P. Rouchon. A simplified approach of crane control via a generalized state-space model. *Proc. IEEE Control and Decision Conference*, pages 736–741, 1991.

[7] M. Fliess, J. Lévine, and P. Rouchon. Genearlized state variable representation for a simplified crane description. *International Journal of Control*, 58(2):277–283, 1993.

[8] J. Hauser, S. Sastry, and G. Meyer. Nonlinear control design for slightly nonminimum phase systems—application to V/STOL aircraft. *Automatica*, 28(4):665–679, 1992.

[9] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse. Systematic design of adaptive controllers for feedback linearizable systems. *IEEE Transactions on Automatic Control*, 36(11):1241–1253, 1991.

[10] J-C. Latombe. *Robot Motion Planning*. Kluwer Academic Publishers, Boston, 1991.

[11] P. Martin. *Contribution a l'Etude des Systems Differentiellement Plats*. PhD thesis, L'Ècole Nationale Supérieure des Mines de Paris, 1992.

[12] P. Martin. Endogenous feedbacks and equivalence. In *Mathematical Theory of Networks and Systems*, Regensburg, Germany, August 1993.

[13] P. Martin. Personal communication, 1994.

[14] P. Martin, S. Devasia, and B. Paden. A different look at output tracking: Control of a VTOL aircraft. In *Proc. IEEE Control and Decision Conference*, pages 2376–2381, 1994.

[15] P. Martin and P. Rouchon. Feedback linearization and driftless systems. *Mathematics of Control, Signals, and Systems*, 7(3):235–254, 1994.

[16] P. Martin and P. Rouchon. Any (controllable) driftless system with 3 inputs and 5 states is flat. *Systems and Control Letters*, 25(3):167–173, 1995.

[17] R. M. Murray. Nilpotent bases for a class of nonintegrable distributions with applications to trajectory generation for nonholonomic systems. *Mathematics of Control, Signals, and Systems*, 7:58–75, 1994.

[18] R. M. Murray. Nonlinear control of mechanical systems: A Lagrangian perspective. In *IFAC Symposium on Nonlinear Control Systems Design (NOLCOS)*, 1995.

[19] R. M. Murray. Trajectory generation for a towed cable flight control system with input constraints using differential flatness. In *1996 IFAC World Congress*, 1996. (submitted).

[20] R. M. Murray and S. S. Sastry. Nonholonomic motion planning: Steering using sinusoids. *IEEE Transactions on Automatic Control*, 38(5):700–716, 1993.

[21] M. Rathinam and W. Sluis. A test for differential flatness by reduction to single input systems. Technical Report CIT/CDS 95-018, California Institute of Technology, 1995.

[22] P. Rouchon, M. Fliess, J. Levine, and P. Martin. Flatness and motion planning: the car with n trailers. In *Proc. European Control Conference*, pages 1518–1522, 1992.

[23] W. M. Sluis. *Absolute Equivalence and its Applications to Control Theory*. PhD thesis, University of Waterloo, 1992.

[24] O. J. Sørdalen. Conversion of the kinematics of a car with N trailers into a chained form. In *Proc. IEEE International Conference on Robotics and Automation*, pages 382–387, 1993.

[25] D. Tilbury, R. M. Murray, and S. S. Sastry. Trajectory generation for the N-trailer problem using Goursat normal form. *IEEE Transactions on Automatic Control*, 40(5):802–819, 1995.

[26] M. van Nieuwstadt and R. M. Murray. Approximate trajectory generation for differentially flat systems with zero dynamics. In *Proc. IEEE Control and Decision Conference*, 1995. (to appear).

[27] M. van Nieuwstadt, M. Rathinam, and R. M. Murray. Differential flatness and absolute equivalence. In *Proc. IEEE Control and Decision Conference*, 1994. Also available as Caltech technical report CIT/CDS 94-006. Submitted, *SIAM J. Control and Optimization*.