

# CDS

TECHNICAL MEMORANDUM NO. CIT-CDS 95-018  
June 1995

**“A Test for Differential Flatness by  
Reduction to Single Input Systems”**

Muruhan Rathinam and Willem M. Sluis

**Control and Dynamical Systems**  
California Institute of Technology  
Pasadena, CA 91125

# A Test for Differential Flatness by Reduction to Single Input Systems

Muruhan Rathinam<sup>1</sup>  
Applied Mathematics  
California Institute of Technology  
Mail Code 217-50  
Pasadena, CA 91125  
muruhan@ama.caltech.edu

Willem M. Sluis  
Control and Dynamical Systems  
California Institute of Technology  
Mail Code 104-44  
Pasadena, CA 91125  
sluis@indra.caltech.edu

Technical Report: CDS95-018

22 June 1995

## Abstract

For nonlinear control systems ( $p$  inputs), we present a test for flatness. The method consists of making an initial guess for  $p - 1$  of the flat outputs, which may involve parameters still to be determined. A choice of functions of time for the  $p - 1$  outputs reduce the system to one with a single input. For single input systems the problem of flatness has been solved and thus leads to the identification of the last flat output, or to obstructions to flatness under the hypotheses. We demonstrate the method for a coupled rigid body in  $\mathbb{R}^2$  and for a single rigid body in  $\mathbb{R}^3$ .

## Keywords

Nonlinear Control, Differential Flatness, Pfaffian System, Feedback Linearization, Cartan Prolongation

AMS Subject Classification: 93C10, 93B29, 58A15, 58A17, 58A10

---

<sup>1</sup>Research supported in part by a grant from the Powell Foundation and by NSF Grant CMS-9502224.

## 1 Introduction

Differential flatness is an important concept in the theory of underdetermined systems of ordinary differential equations. Roughly speaking, a system

$$F^k(t, x^1, \dots, x^N, \dot{x}^1, \dots, \dot{x}^N) = 0 \quad k = 1, \dots, n < N,$$

is differentially flat if there is a smooth 1-1 correspondence between solutions  $x(t)$  of the system and arbitrary functions  $y(t)$ , where  $(y^1, \dots, y^p) \in \mathbb{R}^p$  ( $p = N - n$ ), of the form

$$\begin{aligned} x(t) &= g(t, y(t), \dots, y^{(l)}(t)), \\ y(t) &= h(t, x(t), \dots, x^{(q)}(t)). \end{aligned}$$

Here  $g, h$  are smooth maps and  $l, q$  are integers. The  $y^{(k)}$  is  $k^{\text{th}}$  derivative of  $y$ . The variables  $y^j$  are referred to as flat outputs. The special class of systems given by

$$\dot{x}^i = f^i(t, x^1, \dots, x^n, u^1, \dots, u^p), \quad i = 1, \dots, n$$

are more familiar to control theorists and the flat outputs depend on states, inputs and derivatives of inputs

$$y^j = h^j(t, x, u, u^{(1)}, \dots, u^{(q)}), \quad j = 1, \dots, p.$$

The term differential flatness was coined and introduced by Fliess et al. They initially used differential algebra as a tool to define and study differential flatness, see [3, 1]. Later, they introduced differential flatness in the setting of Lie Bäcklund mappings on infinite jet spaces, see [2], which also allowed them to define “orbital flatness”, a concept more general than differential flatness. See also [6] for a related approach.

Differential flatness was introduced in the framework of exterior differential systems and Cartan prolongations by van Nieuwstadt et al. in [10]. In this paper we shall use the same framework, except that we keep time as a special variable. Hence all transformations are expected to keep time unchanged. We shall not consider orbital flatness and the term “flat” shall stand for differential flatness.

The importance of flatness to control applications lies in the fact that it provides a systematic and relatively simple way to generate solution trajectories between two given states. One uses the maps  $g, h$  to transform between original system space (states as well as inputs) and the smaller dimensional flat output space. See [4] for more details.

















































